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ISABELLE CHALENDAR AND WILLIAM T. ROSS

Abstract. We give a characterization of the compact operators on a model space in terms of asymptotic Toeplitz operators.

1. Introduction

If H^2 denotes the classical Hardy space of the open unit disk $\mathbb{D}$ [4, 7], a theorem of Brown and Halmos [3] says that a bounded linear operator T on H^2 is a Toeplitz operator if and only if

$$S^* T S = T,$$

where $Sf = zf$ is the well-known unilateral shift on H^2. By a *Toeplitz operator* [2], we mean, for a given symbol $\varphi \in L^\infty(\mathbb{T}, m)$ ($\mathbb{T}$ is the unit circle and m is normalized Lebesgue measure on $\mathbb{T}$), the operator

$$T_\varphi : H^2 \to H^2, \quad T_\varphi f = P(\varphi f),$$

where P is the orthogonal projection of L^2 onto H^2.

This notion of "Toeplitzness" was extended in various ways. Barria and Halmos [1] examined the so-called *asymptotically Toeplitz operators* operators T on H^2 for which the sequence of operators

$$\{S^{*n} T S^n\}_{n \geqslant 1}$$

converges strongly. This class certainly includes the Toeplitz operators but also includes other operators such as those in the Hankel algebra. Feintuch [5] discovered that one need not restrict to strong convergence of $\{S^{*n} T S^n\}_{n \geqslant 1}$ and worthwhile classes of operators arise from the weak and uniform (or norm) limits of this sequence. Indeed, an operator T on H^2 is uniformly asymptotically Toeplitz, i.e., $S^{*n} T S^n$ converges in operator norm, if and only if

$$T = T_1 + K, \tag{1.1}$$

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where T_1 is a Toeplitz operator, i.e., $S^*T_1S = T_1$, and K is a compact operator on H^2. Nazarov and Shapiro [8] examined other associated notions of "Toeplitzness" with regards to certain composition operators on H^2.

In this paper we explore a model space setting for this "Toeplitzness" discussion. For an inner function Θ on $\mathbb{D}$ (i.e., a bounded analytic function on $\mathbb{D}$ whose radial boundary values are unimodular almost everywhere on $\mathbb{T}$), one can define the *model space* [6, 9]

$$\mathcal{K}_\Theta = H^2 \ominus \Theta H^2.$$

Beurling's theorem [4] says that these spaces are the generic invariant subspaces for the backward shift operator

$$S^*f = \frac{f - f(0)}{z}$$

on H^2. By model theory for contractions [9], certain types of Hilbert space contractions are unitarily equivalent to compressed shifts

$$S_\Theta = P_\Theta S|_{\mathcal{K}_\Theta},$$

where P_Θ is the orthogonal projection of L^2 onto $\mathcal{K}_\Theta$.

In this model spaces setting, we examine, for a bounded operator A on $\mathcal{K}_\Theta$, the sequence

$$\{S_\Theta^{*n} A S_\Theta^n\}_{n \geqslant 1}.$$

Here we have a similar result as before (see Lemma 2.6 below) in that $S_\Theta^{*n} A S_\Theta^n$ converges in operator norm if and only if

$$A = A_1 + K,$$

where K is a compact operator on $\mathcal{K}_\Theta$ and A_1 satisfies $S_\Theta^* A_1 S_\Theta = A_1$. In the analogous H^2 setting, the operator T_1 from (1.1) is a Toeplitz operator. In the model space setting, the corresponding operator A_1 is severely restricted. Indeed,

$$A_1 \equiv 0.$$

Thus, as the main theorem of this paper, we have the following characterization of the compact operators on $\mathcal{K}_\Theta$.

Theorem 1.2. *For an inner function Θ and a bounded linear operator A on $\mathcal{K}_\Theta$, the following are equivalent:*

(i) The sequence $S_\Theta^{*n} A S_\Theta^n$ converges in operator norm;

(ii) $S_\Theta^{*n} A S_\Theta^n \to 0$ in operator norm;

(iii) A is a compact operator.

One can also explore the convergence of the sequence $S_\Theta^{*n} A S_\Theta^n$ in other topologies, such as the strong/weak operator topologies. Surprisingly what happens is entirely different from what happens in H^2.

Proposition 1.3. *For any inner function Θ and any bounded linear operator A on $\mathcal{K}_\Theta$, the sequence $S_\Theta^{*n} A S_\Theta^n$ converges to zero strongly.*

In other words, the convergence of $S_\Theta^{*n} A S_\Theta^n$ in the strong or weak topology is always true (and to the same operator) and provides no information about A.

2. Characterization of the compact operators

The following lemma proves the implication $(iii) \implies (ii)$ of Theorem 1.2.

Lemma 2.1. *If K is a compact operator on $\mathcal{K}_\Theta$ then*

$$\lim_{n\to\infty} \|S_\Theta^{*n} K S_\Theta^n\| = 0.$$

Proof. Let $\mathcal{B}_\Theta = \{f \in \mathcal{K}_\Theta : \|f\| \leqslant 1\}$ denote the closed unit ball in $\mathcal{K}_\Theta$. First observe that

$$\|S_\Theta^n\| \leqslant \|S_\Theta\|^n \leqslant \|P_\Theta S|_{\mathcal{K}_\Theta}\|^n \leqslant \|S\|^n = 1.$$

From here we see that

$$\begin{aligned} \|S_\Theta^{*n} K S_\Theta^n\| &= \sup_{f\in\mathcal{B}_\Theta} \|S_\Theta^{*n} K S_\Theta^n f\| \\ &\leqslant \sup_{g\in\mathcal{B}_\Theta} \|S_\Theta^{*n} K g\| \\ &\leqslant \sup_{h\in\overline{K(\mathcal{B}_\Theta)}} \|S_\Theta^{*n} h\|. \end{aligned} \tag{2.2}$$

Second, note that $S^{*n} \to 0$ strongly. Indeed, if $f = \sum_{k\geqslant 0} a_k z^k \in H^2$, then

$$\|S^{*n} f\|^2 = \sum_{k\geqslant n+1} |a_k|^2 \to 0 \quad n \to \infty.$$

Thus since $S_\Theta^{*n} = S^{*n}|_{\mathcal{K}_\Theta}$ (since $\mathcal{K}_\Theta$ is S^*-invariant), we see that

$$S_\Theta^{*n} \to 0 \text{ strongly.} \tag{2.3}$$

Let $\epsilon > 0$ be given and let $h \in \overline{K(\mathcal{B}_\Theta)}$. Since $S_\Theta^{*n} \to 0$ strongly, there exists an $n_{h,\epsilon}$ such that $\|S_\Theta^{*n} h\| < \epsilon/2$ for all $n > n_{h,\epsilon}$. The continuity of the operator $S^{*n_{h,\epsilon}}$ implies that there exists a $r_{h,\epsilon}$ such that for all q belonging to

$$B(h, r_{h,\epsilon}) = \{q \in \mathcal{K}_\Theta : \|q - h\| < r_{h,\epsilon}\}$$

we have $\|S^{*n_{h,\epsilon}} q\| < \epsilon$.

Again using the fact that $\|S_\Theta^*\| \leqslant 1$, we see that for all $q \in B(h, r_{h,\epsilon})$ and all $n > n_{h,\epsilon}$ we have

$$\|S_\Theta^{*n} q\| = \|S_\Theta^{*(n-n_{h,\epsilon})} S_\Theta^{*n_{h,\epsilon}} q\| \leqslant \|S_\Theta^{*n_{h,\epsilon}} q\| < \epsilon. \tag{2.4}$$

Moreover, we have

$$\overline{K(\mathcal{B}_\Theta)} \subset \bigcup_{h \in \overline{K(\mathcal{B}_\Theta)}} B(h, r_{h,\epsilon}).$$

The compactness of $\overline{K(\mathcal{B}_\Theta)}$ implies that there exists $h_1, \ldots, h_N$ ($N = N_\epsilon$) belonging to $\overline{K(\mathcal{B}_\Theta)}$ such that

$$\overline{K(\mathcal{B}_\Theta)} \subset \bigcup_{k=1}^{N} B(h_k, r_{h_k,\epsilon}).$$

For all $n > \max\{n_{h_1,\epsilon}, \ldots, n_{h_N,\epsilon}\}$ we use (2.4) along with (2.2) to see that

$$\|S_\Theta^{*n} h\| < \epsilon \quad \forall h \in \overline{K(\mathcal{B}_\Theta)}.$$

This proves the lemma. □

Remark 2.5. Important to the proof above was the fact that $S_\Theta^{*n} \to 0$ strongly (see (2.3)). One can show that S_Θ is unitarily equivalent to S_Ψ^*, where Ψ is the inner function defined by $\Psi(z) = \overline{\Theta(\overline{z})}$ [6, p. 303]. From here we see that $S_\Theta^n \to 0$ strongly. This detail will be important at the end of the paper in the proof of Theorem 1.3.

Lemma 2.6. *Suppose T is a bounded operator on $\mathcal{K}_\Theta$ such that $S_\Theta^{*n} T S_\Theta^n$ converges in norm. Then $T = T_1 + K$, where K is a compact operator on $\mathcal{K}_\Theta$ and T_1 is a bounded operator on $\mathcal{K}_\Theta$ satisfying $S_\Theta^* T_1 S_\Theta = T_1$.*

Proof. Let A be a bounded operator on $\mathcal{K}_\Theta$ such that

$$\|S_\Theta^{*n} T S_\Theta^n - A\| \to 0.$$

Then

$$\begin{aligned} \|S_\Theta^{*(n+1)} T S_\Theta^{n+1} - S_\Theta^* A S_\Theta\| &= \|S_\Theta^* (S_\Theta^{*n} T S_\Theta^n - A) S_\Theta\| \\ &\leqslant \|S_\Theta^{*n} T S_\Theta^n - A\| \to 0. \end{aligned}$$

This implies that

$$S_\Theta^* A S_\Theta = A. \tag{2.7}$$

From here it follows that

$$S_\Theta^{*n} T S_\Theta^n - A = S_\Theta^{*n}(T - A)S_\Theta^n, \quad n \geqslant 0. \tag{2.8}$$

Define

$$P_n := S_\Theta^n S_\Theta^{*n} \text{ and } Q_n := I - P_n = I - S_\Theta^n S_\Theta^{*n}$$

and observe that

$$\begin{aligned} P_n(T - A)P_n = (T - A) \\ - Q_n(T - A) + Q_n(T - A)Q_n - (T - A)Q_n. \end{aligned} \tag{2.9}$$

Furthermore by (2.8) we have

$$\begin{aligned} \|P_n(T - A)P_n\| &= \|S_\Theta^n S_\Theta^{*n}(T - A)S_\Theta^n S_\Theta^{*n}\| \\ &\leqslant \|S_\Theta^{*n}(T - A)S_\Theta^n\| \\ &= \|S_\Theta^{*n} T S_\Theta^n - A\| \to 0. \end{aligned}$$

If

$$k_\lambda(z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \overline{\lambda} z}, \quad \lambda, z \in \mathbb{D},$$

is the reproducing kernel for $\mathcal{K}_\Theta$, then [10, p. 497] gives us the well-known operator identity

$$S_\Theta S_\Theta^* = I - k_0 \otimes k_0.$$

Iterating the above n times we get

$$S_\Theta^n S_\Theta^{*n} = I - \sum_{j=0}^{n-1} S_\Theta^j k_0 \otimes S_\Theta^{*j} k_0.$$

In other words,

$$Q_n = I - P_n = \sum_{j=0}^{n-1} S_\Theta^j k_0 \otimes S_\Theta^{*j} k_0$$

is a finite rank operator.

By (2.9) this means that

$$F_n := -Q_n(T - A) + Q_n(T - A)Q_n - (T - A)Q_n$$

is a finite rank operator which converges in norm to $A - T$. Hence $A - T$ a compact operator and, by (2.7), A satisfies $S_\Theta^* A S_\Theta = A$. $\square$

So far we know from Lemma 2.1 that every compact operator K on $\mathcal{K}_\Theta$ satisfies

$$\lim_{n\to\infty} \|S_\Theta^{*n} K S_\Theta^n\| = 0.$$

Furthermore, from Lemma 2.6 we see that an operator A for which $S_\Theta^{*n} A S_\Theta^n$ converges in operator norm can be written as $A = A_1 + K$ where K is compact and A_1 satisfies $S_\Theta^* A_1 S_\Theta = A_1$. To complete the proof of Theorem 1.2, we need to show that

$$S_\Theta^* A S_\Theta = A \iff A \equiv 0.$$

This is done with the following result.

Proposition 2.10. *Suppose A is a bounded operator on $\mathcal{K}_\Theta$. Then $S_\Theta^* A S_\Theta = A$ if and only if $A \equiv 0$.*

Proof. Recall that

$$k_\lambda(z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \overline{\lambda} z}$$

is the kernel function for $\mathcal{K}_\Theta$. There is also the "conjugate kernel"

$$\widetilde{k}_\lambda(z) = \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda}$$

which also belongs to $\mathcal{K}_\Theta$ [10, p. 495]. The proof depends on the following kernel function identities from [10, p. 496]:

$$S_\Theta \widetilde{k}_\lambda = \lambda \widetilde{k}_\lambda - \Theta(\lambda) k_0,$$

$$S_\Theta k_\lambda = \frac{1}{\overline{\lambda}} k_\lambda - \frac{1}{\overline{\lambda}} k_0.$$

This gives us

$$\begin{aligned}
(A\widetilde{k}_\lambda)(z) &= \langle S_\Theta^* A S_\Theta \widetilde{k}_\lambda, k_z\rangle \\
&= \langle A S_\Theta \widetilde{k}_\lambda, S_\Theta k_z\rangle \\
&= \langle A(\lambda \widetilde{k}_\lambda - \Theta(\lambda)k_0), \frac{1}{\overline{z}} k_z - \frac{1}{\overline{z}} k_0\rangle \\
&= \frac{\lambda}{z}(A\widetilde{k}_\lambda)(z) - \frac{\Theta(\lambda)}{z}(Ak_0)(z) - \frac{\lambda}{z}(A\widetilde{k}_\lambda)(0) + \frac{\Theta(\lambda)}{z}(Ak_0)(0).
\end{aligned}$$

Re-arrange the above identity:

$$(A\widetilde{k}_\lambda)(z)(1 - \frac{\lambda}{z}) = -\frac{\Theta(\lambda)}{z}(Ak_0)(z) - \frac{\lambda}{z}(A\widetilde{k}_\lambda)(0) + \frac{\Theta(\lambda)}{z}(Ak_0)(0).$$

Multiply through by z:

$$(z - \lambda)(A\widetilde{k}_\lambda)(z) = -\Theta(\lambda)(Ak_0)(z) - \lambda(A\widetilde{k}_\lambda)(0) + \Theta(\lambda)(Ak_0)(0).$$

Divide by $(z-\lambda)$ and re-arrange:

$$(A\widetilde{k_\lambda})(z) = -\Theta(\lambda)\left(\frac{(Ak_0)(z)-(Ak_0)(\lambda)}{z-\lambda}\right) - \lambda\frac{(A\widetilde{k_\lambda})(0)}{z-\lambda}. \tag{2.11}$$

Observe that the functions

$$(A\widetilde{k_\lambda})(z) \text{ and } \frac{(Ak_0)(z)-(Ak_0)(\lambda)}{z-\lambda}$$

belong to $\mathcal{K}_\Theta$ for all $\lambda \in \mathbb{D}$. This means that

$$\lambda\frac{(A\widetilde{k_\lambda})(0)}{z-\lambda}$$

must also belong to $\mathcal{K}_\Theta$ for all $\lambda \in \mathbb{D}$ which means (since there is an obvious pole at $z=\lambda$) that

$$(A\widetilde{k_\lambda})(0) = 0. \tag{2.12}$$

The identity in (2.11) can now be written as

$$(A\widetilde{k_\lambda})(z) = -\Theta(\lambda)\left(\frac{(Ak_0)(z)-(Ak_0)(\lambda)}{z-\lambda}\right). \tag{2.13}$$

Plug in $z=0$ into the previous identity and use (2.12) to see that

$$0 = (A\widetilde{k_\lambda})(0) = \frac{\Theta(\lambda)}{\lambda}((Ak_0)(0)-(Ak_0)(\lambda)), \quad \lambda \in \mathbb{D}.$$

Since Θ is not the zero function, we get

$$(Ak_0)(\lambda) = (Ak_0)(0), \quad \lambda \in \mathbb{D}. \tag{2.14}$$

Plus this into (2.13) to get that

$$A\widetilde{k_\lambda} = 0 \quad \forall \lambda \in \mathbb{D}.$$

But since the linear span of these conjugate kernels form a dense subset in $\mathcal{K}_\Theta$ (the conjugation operator $f \mapsto \widetilde{f}$ is isometric and involutive [10, p. 495]), we see that A must be the zero operator. □

Proof of Proposition 1.3. For any $f, g, \in \mathcal{K}_\Theta$ and $n \geqslant 0$ we have

$$|\langle S^{*n}AS^n f, g\rangle| = |\langle S_\Theta^n f, A^* S_\Theta^n g\rangle| \leqslant \|S_\Theta^n f\| \|A^* S_\Theta^n g\|. \tag{2.15}$$

Taking the supremum in (2.15) over $g \in \mathcal{K}_\Theta$ with $\|g\| \leqslant 1$, and using the fact that $\|S_\Theta\| \leqslant 1$, we get

$$\|S^{*n}AS^n f\| \leqslant \|S_\Theta^n f\| \|A^*\|. \tag{2.16}$$

From Remark 2.5, we conclude that the right hand side of (2.15) goes to zero as $n \to \infty$. Thus $S_\Theta^{*n}AS_\Theta^n \to 0$ strongly. □

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